Mathematical Modeling in Action

One of the most beautiful features of applied linear algebra is the deep connection between practical applications and mathematical theory. Introductory courses in linear algebra provide a natural entry point for you to explore significant mathematical questions while engaging in applied mathematical modeling to study real-world problems related to your majors. Indeed, linear algebra plays a prominent role in undergraduate STEM curriculum and "is essential for advanced work in the sciences, statistics, and computing" [CUPM15LA]. Decades of advancements in computer hardware and software have positioned linear algebra as a fundamental tool in solving diverse problems in electrical engineering [Vlach1994], mechanical and civil engineering [Craig2006, Logan2012], computer graphics [Farin2014], data science [Strang2019], statistics [Gentle2017], machine learning [Aggarwal2020], image processing [Ryan2019], operations research [Vanderbei2013, Nocedal2006], economics [Aleskerov2011], financial engineering [Stefanica2014], robotics [Corke2017, Scaglia2020], data mining and pattern recognition [Elden2019], internet search [Langville2006], chemistry [Fleisch2020], and other STEM fields. This theory forms a foundational pillar of modern applied mathematics and mathematical modeling.

However, it is challenging to convey the importance of this subject to students in introductory classes. This is partly because traditional introductory linear algebra textbooks focus heavily on abstract theory that is decontextualized from students' interest. Too often, conventional curricula do not empower students to collect data from the material world, do not engage students in applied modeling activities, and do not invite students to solve real-world problems that relate to their majors. Students often finish introductory linear algebra classes unprepared to transfer their knowledge and less-than-interested in continuing their studies of this field.

Luckily, this is not a traditional textbook. This textbook is designed to enrich your exploration of linear algebra with a variety of real-world problems, authentic modeling activities, and fun laboratory explorations designed to ignite your interest in this subject. Together with your classmates, you will study problems that are directly related to your academic and career interests, build transferable skills that help give you a competitive advantage in your upper-division courses, engage with the material world by completing meaningful laboratory experiences, and build authentic linear algebraic problems from real-world phenomena that you care about. The goal of this work is to help you explore linear algebra not by forcing you to comply with harsh grading schemes, out-of-touch lectures, and high-pressure in-class exams but instead by inviting you to experience the power of linear algebra in action. You will be in the driver's seat as you use linear algebra to model real-world phenomena directly related to your major.

Before we dive into specific real-world problems that can be modeled and studied using linear algebra, let's zoom out a little and ask ourselves some big questions including:

- Why does our society put such a heavy emphasis on teaching mathematics?
- Why is this linear algebra course a required part of your college degree?
- How can you use the skills you build in this class to propel your career and solve problems that you care about?

These questions all point towards the reality that mathematics, and specifically linear algebra, is a powerful tool that you can use to understand and deal with your observable world. You can use mathematics to ask and answer questions that you find meaningful and to enhance your future. You can use mathematics to achieve financial independence and to create a career you love.

To accomplish these goals, you deserve the opportunity to learn and appreciate mathematical modeling in action. Following the Society of Industrial and Applied Mathematics (SIAM) Guidelines for Assessment and Instruction in Mathematical Modeling Education [GAIMME2019, p. 8], let's define mathematical modeling.

Definition: Mathematical Modeling

Mathematical modeling is a process that uses mathematics to represent, analyze, make predictions about, or otherwise provide insight into real-world phenomena.

In this text, we delineate seven unique steps within this larger mathematical modeling process, as highlighted in Figure 0.1 below.

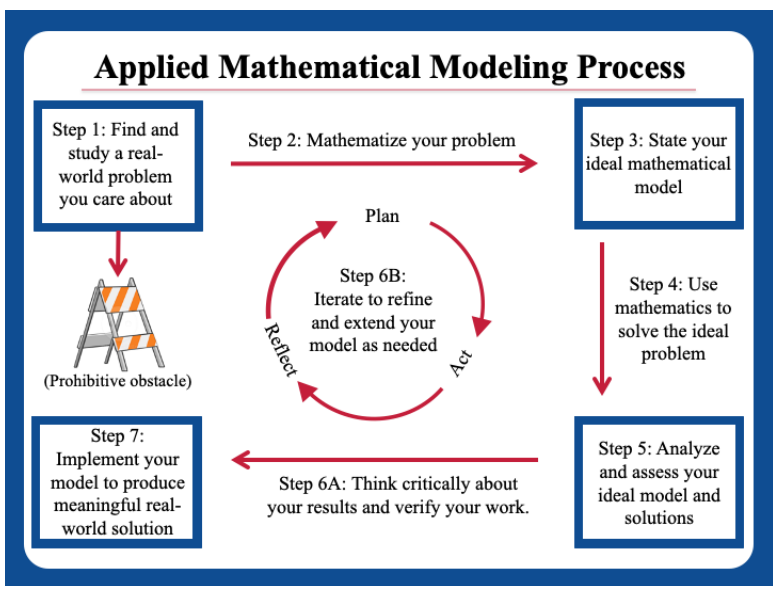

Figure 0.1: The 7 steps of the applied mathematical modeling process.

In this lesson, we describe these steps in detail. After you understand the broad contours of this process, you will choose a laboratory explorations that highlights a problem that is directly related to your major. As you complete your lab, you will get practice with each step and combine these together to complete the entire mathematical modeling process. We've put significant effort into choosing rich-interdisciplinary problems that are accessible to novice students.

Each laboratory experience satisfies many criteria for good interdisciplinary problems [Reinholz2018]. These projects are accessible, offering low-floor problems that are easy for you to start. You only need access to the laboratory resources and project prompts. Once you have these resources, you can walk through the modeling process to create authentic linear algebra problems from observable real-world phenomena. Each laboratory activity is designed to generalize to harder tasks that relate to your future academic and career interests. To analyze each model, you and a small group of your teammates can use multiple paths to solve your problem. Then, you compare the various approaches your team came up with to develop deeper knowledge about the underlying modeling context. By completing each laboratory activity, you get insights into core linear-algebraic concepts like matrix-matrix multiplication, linear-systems problems, least-squares problems, or eigenvalues problems. The mathematical technologies needed in each laboratory experience are exactly the mathematical concepts you study in this book and are standard linear algebra curricula. Each project is context-rich and related to real-world STEM problems. These problems are rich resources aimed at bringing this introductory class alive.

We'll dive deeper into the laboratory explorations in future lessons. For now, let's explore each step of this process individually. Then we'll think about how and where linear algebra can be used to develop meaningful mathematical models within this framework.

Step 1: Find and study a real-world problem you care about

The first phase of this process begins by identifying a *real-world problem*, which we define as any problem that matters in your life and that can be studied by making observations and collecting measured data. One of the fascinating features of real-world problems is that by identifying and imagining the problem, we likely develop some idea of what a meaningful solution might be. However, from the standpoint of applied mathematics, a valuable real-world problem usually includes major obstacles that block the path between the problem statement and our desired solution. In order to earn healthy paychecks, applied mathematicians pray that such obstacles are so prohibitive that spending months or years doing tedious mathematical analysis proves to be much more productive and economical than trying to solve the problem using brute force.

As you become a more sophisticated scientific, mathematical, and entrepreneurial thinker, you will learn to imagine real-world problems that you can study using mathematics. One of the goals of this book is to help you develop your ability to use linear algebra to solve practical problems in your world. In the early stages of this work, it can be very helpful to engage with explicit examples of problems that relate to your major and can be solved using the linear algebraic theory you study in this course. In this textbook, we provide you with a library of rich problems, in the form of laboratory explorations, that are designed to do exactly this. Each laboratory exploration consists of a sequence of open-ended questions and active learning activities that you complete with your classmates in small groups.

Step 2: Mathematize the problem

Once we've identified a real-world problem in which mathematical modeling seems more hopeful than physical labor, we move on to the second phase of the applied mathematical modeling process. In this stage, we *mathematize* our real-world problem by transforming our observations, measurable data, and nonmathematical objects into a collection of relevant mathematical ideas. This might include introducing useful mathematical notation, defining relevant variables, imposing appropriate mathematical assumptions, and focusing on a subset of important problem characteristics while ignoring other aspects of the problem.

As you grow your skills, you will level up to more interesting scientific and industrial problems that likely require specialized training to study, understand, and solve. For good applied mathematics problems, interdisciplinary teams of scientists, engineers, and business professionals can prove to be invaluable. Each professional is trained in a particular field and brings that expertise into the team to propel progress and inform group decision making processes. By combining talents with a growth mindset and some luck, great learning and group work can result in a useful modeling scheme that solves the original problem. One of the goals of our laboratory experiences is to give you practice with authentic experiences working with others that replicate the type of work environments you are likely to encounter in your future academic and career pursuits.

Step 3: State your ideal mathematical model

The next step in the mathematical modeling process is to state your ideal mathematical model. In linear algebra, such ideal models come in a variety of forms which we call the *major problems of linear algebra*. These major problems come in five main flavors including:

Problem type 1: Matrix multiplications problems

Problem type 2: Linear systems problems

Problem type 3: Least squares problems

Problem type 4: Eigenvalue problems

Problem type 5: Advanced linear algebra problems

Each problem type provides a library of common mathematical structures you can use to state modeling problems. You might think of each major problem in linear algebra as a template. The goal of applied linear algebra is to transform some real-world problem into one of the major problems of linear algebra and then use the mathematical content of this course to gain deep insights into the underlying phenomena. The entire study of linear algebra involves learning the mathematical theory and techniques you can use to solve problems that fit into one of these famous problem types. Later in this section, we explore the major problems and delineate various subproblems within each type. The goal of any good modeling process that leverages linear algebra is to transform your real-world problem into one of the templates provided by the four major problems in linear algebra.

In general, the process of creating a useful mathematical model from a real-world problem requires deep thinking, hard work, and a lot of creative thinking. This goes beyond making assumptions and introducing variables. Often this work requires significant mathematical analysis to transform the physical problem into a known mathematical structure that fits into one of the five problem types highlighted in this section. This part of the process also requires deep technical expertise and guidance to ensure the mathematical statement captures the key dynamics of the underlying real-world problem. Because our society does not fund education adequately, traditional teaching and learning experiences in many schools are devoid of meaningful experiences for students to learn how to do this. In this textbook, we fill in the gap by walking you through the process of creating linear algebraic problems from contexts that you care about. As you complete your laboratory experience, you will grow your ability to transform real-world problems into ideal models that fit into one of the five major problems linear algebra. You will then analyze and solve these problems using linear algebraic theory and techniques.

Step 4: Use mathematics and computation to solve your ideal problem

After you transform your real-world problem into one of the major linear algebraic problems, you'll ask yourself the questions "where can I find some mathematics to help me with this problem?" and "what mathematical or computations tools are best suited to help me make progress towards a solution?" To answer these questions, you'll need to decide on which mathematical or computational approaches you feel is most productive. This process often involves deep thinking about the mathematical objects you use in your model, the assumptions that produced those mathematical objects, and the structure of the problem that results from these choices. Such analysis also includes a search for the most powerful mathematical theorems and tools that can be brought to bear on our linear algebraic problem to produce a desired solution.

The mathematical and computational work involved in this step of the process includes many different types of thinking. To solve your problem, you may need to do a literature review to find and apply appropriate formulae from various fields of study that relate to your problem. You may also need to algebraically manipulate or simplify relevant formulae to address the various constraints of the model you've created. The history of digital computers is intertwined with the desire to create computing machines that can be used to do calculations that help solve complex mathematical problems. Thus, you may choose to use technology to do calculations, produce graphical representations of the model, and verify algebraic results. One of the reasons that linear algebra is such a powerful tool in mathematical modeling is that this theory sits at the intersection of mathematics, computations, and applied modeling. For every theorem you study and technique you learn, there are entire software libraries available to leverage these techniques on modern computers and do calculations a lightening fast speeds.

For a good real-world problem, this type of applied mathematical analysis requires many hours of iterated failure until we converge on the most potent mathematical ideas. The hope of applied mathematics is that we might analyze our ideal model using a suite of technical mathematical results and produce an ideal solution to this problem. The dream of applied mathematics is that the ideal solution we produce via mathematical analysis leads to valuable progress in discovering aspects of the meaningful real-world solution that we so desire.

This step is the usual target of traditional mathematics courses. Many math teachers design their class simply to teach the math you'll need to solve problems. In doing so, they hide all interesting aspects of the larger modeling process from students and spend very little time motivating why the math is powerful. This decontextualized approach to teaching mathematics places a larger burden on students' shoulders to figure out how the math you study can be used in the larger world. In this work, we combat this traditional approach by empowering you to experience the entire process. The goal of each laboratory experience is to help you discover how the math you study can be used in the context of solving real-world problems. Once you have that insight, you'll find that the more you know about the math, the more power you'll have in deciding which theorems, algorithms, and techniques are best suited to solve your technical problems.

Step 5: Analyze and asses your ideal model and solutions

Once you've produced your ideal solution, you'll want to analyze these results to get them into a form that makes the most sense for your problem. You'll likely need to dedicate time and effort to understanding what your results mean and to make explicit connections between your solution and your model. To do this, you might use a series of questions about your model. Do these results make sense? How accurate and precise are my solutions? Do my interim or final results aligned with what I expect? Are any of my interim or final results unexpected? Do my results align with and satisfy the constraints I've imposed in my models? In what contexts does this model provide valuable information? How much do the solutions change if I vary the assumptions, model, or parameters? Would it improve the model to change something decided at an earlier step and resolve the problem?

The major goal of this step of the process is to fully understand and make sense of the mathematical solutions you've produced. Remember, the solution to your problem comes from your analysis of your ideal mathematical model. Every component of your model comes from simplifying assumptions, variable choices, and imposed constraints you introduced when transforming your real-world problem into your ideal model. Thus, in this step, you're making explicit connections between your model and your solution to ensure that you understand what your solutions means within the larger context of the modeling process.

Step 6A: Think critically about your results and verify your work

The next step is to figure out the real-world meaning of your solution. To do this, you will map your mathematical results to their real-world counterparts. You will also contextualize your mathematical results in terms of the real-world situation you are studying. How do your calculated solutions and final variable values relate back to the original real-world data you observed? How accurate is your modeled data compared to the measured data you collected when studying the original problem? What discrepancies show up in your ideal solution versus your real-world data? What does this say about your model? Are these differences acceptable or do they point to some larger issue with the way you've completed your work? Have you addressed all your goals in this modeling problem? Have you produced a viable solution? Is your ideal solution adequate to solve the original real-world problem you started solving?

Step 6B: Iterate to refine and extend your model as needed

Statistician George Box famously wrote that "all models are approximations. Essentially, all models are wrong, but some models are useful." He also asked the question "how wrong does a model have to be to be not useful?" These quotes get at a fundamental truth in the mathematical modeling process. In authentic applied modeling contexts, the observations and measured data you generate from studying your real-world problem will not exactly match the ideal solution you produce through the modeling process. However, just because there are differences between your model and your real-world observations does not mean your model is useless. The question is what types of inaccuracies are acceptable and how can you evaluate the trustworthiness of your modeling approach.

It is very often the case that during the modeling process, your first draft results will not be in the exact form you need to solve your larger problem. If you decide that you need to improve your results, you might go back to the drawing board and improve your model. This might involve changing your decisions you made early in the modeling process to introduce new ideas. Or you might need to relax some of the constraints you imposed to simplify your problem. Indeed, the simplicity-first approach implies that in the early stages of modeling, you might make a series of assumptions to lead to a very simple model and help produce a viable first-draft solution. While this solution may be very insightful for getting your head around the larger modeling process, you may want more sophisticated approaches to address the needs of your original problem. It is at this stage of the process that you may want to refine your approach, relax your constraints, and introduce more complexity into your modeling process.

One piece of advice that is useful in this step is: "don't make things more complex until you're sure that you've solved your easiest problems." This is related to a similar idea about optimizing computer software that says: "don't try to optimize your software until you're sure that the code you've written actually solves the problem you're trying to solve."

Step 7: Implement your model to produce real-world solutions

Professional authors and writers often live by the mantra "you're never really done writing, editing, and refining your work. What happens is you do as much as you can and then your deadline hits." The same is true with mathematical modeling. Mathematical modeling is an iterative process. Applied mathematicians can iterate and improve their models indefinitely. But deadlines are real. At some point in your modeling process, it will be time to declare victory or defeat and report your results to the appropriate people. If you're expecting to be paid for your work, this process will involve demonstrating that your work addresses the needs of your funder(s) and solves the problem you were paid to solve. Assuming you've done good work in steps 1 - 6, you should also be able to explain what you might do to improve your model, in what situations your model is and is not useful, and how to leverage your work to make the appropriate decisions. This part of the process often involves presenting your work, discussing the relevant features of your solution, answering technical questions about your approach, and troubleshooting to ensure you've solved the original problem you set out to solve.

The Major Problems of Linear Algebra

Now that we have a general overview of the seven steps of the mathematical modeling process, we're going to explore a high-level overview of exactly where the theory of linear algebra fits into this framework. Notice that in Step 3 of our process, modelers transform the real-world problem into an ideal mathematical model. As mentioned above, linear algebra is the study of a collection of ideal mathematical models that come in special forms. In this introductory textbook, we study five types of ideal models, each one having special sub-types. That is the exact subject of our next lesson.

Suggested Exercises:

1. Look over the list of references given in the first paragraph of this section. These include a number of books that highlight the power and beauty of linear algebra to solve applied problems in diverse STEM fields. Think about your own interests. What type of problems do you want to solve? Why? Now look at the list of resources offered at the end of this section. Choose one or two books and spend 30 minutes researching that book. What did you learn? If you do not find a book on a topic you're interested in on that list, then do some research for yourself to figure out which interests you have that might be related to linear algebra.

2. Choose any one of the five major problems of linear algebra:

   Problem type 1: Matrix multiplications problems

   Problem type 2: Linear systems problems

   Problem type 3: Least squares problems

   Problem type 4: Eigenvalue problems

   Problem type 5: Advanced linear algebra problems

   Now, get ready to do some research online. Set your timer for 30 minutes. Then, without getting distracted, see what you can learn about where where this problem shows up in real-world modeling contexts that are directly related to your academic and career interests. Be sure to track the websites, videos, or resources that you used to find answers to this question. Please share these with your classmates and also with your teacher.

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