

Undergraduate Textbook Draft

Applied Linear Algebra Fundamentals

First Edition

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March 8, 2023

To my loving family: past, present and future

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